Université Abou Bekr Belkaid Tlemcen
Faculté de Technologie
Spécialité Génie Productique

Année 2013-2014
$1^{ère}$ Année
Outils Mathématiques

Examen final

Exercice 1 : (07points)

1. Résoudre l'équation différentielle suivante :

$$y'' + 3y' + 2y = e^{-x}$$

2. Trouver la solution qui vérifie :

$$y(0) = y'(0) = 0$$

Exercice 2 : (07 points)

Calculer ce qui suit :

$$I_1 = \int \frac{3x-1}{x^2+x+5}dx \quad ; \qquad I_2 = \int \frac{x}{x^4+4}dx \quad ; \qquad I_3 = \int (lnx)^2 dx$$

Exercice 3 : (06 points)

Soit $n \in \mathbb{N}, n > 2$, résoudre dans $\mathbb{C}$ l'équation suivante :

$$z^{2n} + z^n + 1 = 0$$

Barème :

Exercice 1 : 7 points ; question1= 4 points, question2=3 points.

Exercice 2 : 7 points ; I1= 3points, I2= 2 points, I3=2points

Exercice 3 : 6 points.

— Outils Mathématiques —

— Corrigé —

Exercice 1 : 1/ $y''+3y'+2y=e^{-x}$

ESSM : $y''+3y'+2y=0$; Equation caractéristique : $r^2+3r+2=0$

$\Delta = 1>0$; $r_1=-2$, $r_2=-1$

La solution générale de l'ESSM est donnée par : $\boxed{y_0 = C_1 e^{-2x} + C_2 e^{-x}}$ $C_1, C_2 \in \mathbb{R}$ 1pt

" " " " l'EASM " " " $y = y_0 + y_*$

$y_* = C_1(x) e^{-2x} + C_2(x) e^{-x}$ (variation de constante).

$$\begin{cases} C_1'(x) e^{-2x} + C_2'(x) e^{-x} = 0 \\ -2C_1'(x) e^{-2x} - C_2'(x) e^{-x} = e^{-x} \end{cases} \Rightarrow C_1'(x) = -e^{+x} \Rightarrow \boxed{C_1(x) = -e^{x}}$$ 0,5 pt

$$\Rightarrow C_2'(x) = 1 \Rightarrow \boxed{C_2(x) = x}$$ 0,5 pt

Ainsi $\boxed{y_* = (x-1)e^{-x}}$ 01pt

$y = y_0 + y_* \Rightarrow \boxed{y = C_1 e^{-2x} + C_2 e^{-x} + (x-1)e^{-x}}$ 01pt

2/ $y(0)=0 \Rightarrow C_1 + C_2 - 1 = 0 \Rightarrow C_1 + C_2 = 1$

$y'(x) = -2C_1 e^{-2x} - C_2 e^{-x} + e^{-x} + (1-x)e^{-x} \Rightarrow [y'(0)=0 \Rightarrow -2C_1 - C_2 + 2 = 0]$

on obtient le système $\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$ 1pt 1pt

Finalement : $\boxed{y = e^{-2x} + (x-1)e^{-x}}$ 1pt

EXERCICE 2 :

1/

$$I_1 = \int \frac{3x-1}{x^2+x+5}\,dx \qquad \Delta = 1-20 = -19 < 0 .$$

$$I_1 = \int \frac{3/2(2x+1)-1-3/2}{x^2+x+5}\,dx = 3/2\int \frac{2x+1}{x^2+x+5}\,dx - 5/2\int\frac{1}{x^2+x+5}\,dx$$

$$I_1 = 3/2\,\ln(x^2+x+5) - 5/2\,J \qquad \text{1pt}$$

$$J = \int \frac{1}{x^2+x+5}\,dx \qquad x^2+x+5 = (x+1/2)^2+5-\frac{1}{4} = (x+1/2)^2+\frac{19}{4}.$$

$$J = \int \frac{1}{(x+1/2)^2+\frac{19}{4}}\,dx = \frac{4}{19}\int \frac{1}{\left(\frac{2x+1}{\sqrt{19}}\right)^2+1}\,dx \quad \text{On pose } t = \frac{2x+1}{\sqrt{19}} \Rightarrow dt = \frac{2}{\sqrt{19}}\,dx \Rightarrow dx = \frac{\sqrt{19}}{2}\,dt$$

$$J = \frac{2}{\sqrt{19}}\int\frac{1}{t^2+1}\,dt \Longrightarrow \boxed{J = \frac{2}{\sqrt{19}}\,\mathrm{Arctg}\left(\frac{2x+1}{\sqrt{19}}\right)} \qquad \text{1pt}$$

Finalement : $\boxed{I_1 = 3/2\,\ln(x^2+x+5) - \frac{5}{\sqrt{19}}\,\mathrm{Arctg}\left(\frac{2x+1}{\sqrt{19}}\right) + C \quad C\in\mathbb{R}} \longrightarrow$ 1pt

2/

$$I_2 = \int \frac{x}{x^4+4}\,dx = \frac{1}{2}\int\frac{2x}{x^4+4}\,dx \quad \text{On pose } t = x^2 \Rightarrow dt = 2x\,dx. \qquad \text{1pt}$$

$$I_2 = 1/2\int\frac{1}{t^2+4}\,dt \Rightarrow I_2 = \frac{1}{8}\int\frac{1}{\left(\frac{t}{2}\right)^2+1}\,dt \quad \text{On pose } u = t/2 \Rightarrow du = 1/2\,dt$$

$$I_2 = 1/4\int\frac{1}{u^2+1}\,du \Rightarrow I_2 = 1/4\,\mathrm{Arctg}\,u + C \Rightarrow \boxed{I_2 = 1/4\,\mathrm{Arctg}\left(\frac{x^2}{2}\right) + C \quad C\in\mathbb{R}} \qquad \text{1pt}$$

3/ $I_3 = \int(\ln x)^2\,dx$: par parties $\begin{cases} f = (\ln x)^2 \longrightarrow f' = 2\cdot\frac{1}{x}\ln x \\ g' = 1 \longrightarrow g = x. \end{cases}$

$$I_3 = x\cdot(\ln x)^2 - \int 2\ln x\,dx \qquad \text{1pt}$$

par parties $\begin{cases} f = \ln x \longrightarrow f' = 1/x \\ g' = 2 \longrightarrow g = 2x \end{cases} \Rightarrow I_3 = x(\ln x)^2 - \left[2x\ln x - \int 2\,dx\right]$ — 0,5pt

$$\boxed{I_3 = x(\ln x)^2 - 2x\ln x + 2x + C \quad C\in\mathbb{R}} \qquad \text{0,5pt}$$

Exercice 3 :

$$z^{2n} + z^n + 1 = 0 \qquad n \in \mathbb{N} \; ; \; n > 2$$

Equation bicarrée ; on pose $z^n = Z$; l'équation devient :

$$Z^2 + Z + 1 = 0 \; ; \qquad \Delta = 1 - 4 = 3i^2.$$

$$Z_1 = \frac{-1 + \sqrt{3}\,i}{2} = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \quad \text{et} \quad Z_2 = \frac{-1 - \sqrt{3}\,i}{2} = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \text{ —— } 1\text{pt}$$

Ainsi les solutions de l'équation : $z^{2n} + z^n + 1 = 0$; sont les racines $n$ièmes de Z_1 et de Z_2.

$$Z_1 = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i = e^{2i\pi/3} \; ; \; Z_2 = \overline{Z_1} = e^{-2i\pi/3} \left(= e^{i4\pi/3} \right) \text{ —— } 1\text{pt}$$

$$\left(re^{i\theta}\right)^n = Z_1 \Rightarrow \begin{cases} r^n = 1 \\ n\theta = \frac{2\pi}{3} + 2k\pi \end{cases} \Rightarrow \begin{cases} r = 1 \\ \theta = \frac{2\pi}{3n} + \frac{2k\pi}{n} = \frac{(6k+2)\pi}{3n} = \frac{(3k+1)2\pi}{3n} \end{cases} \quad k = 0, 1, 2, \ldots, (n-1)$$

$$z_1 = e^{i\frac{2\pi}{3n}} \; ; \; z_2 = e^{i\frac{8\pi}{3n}}, \ldots, \; z_n = e^{i\frac{(3n-2)2\pi}{3n}} \text{ —— } 2\text{pts}$$

$$\left(re^{i\theta}\right)^n = Z_2 = \begin{cases} r^n = 1 \\ n\theta = -\frac{2\pi}{3} + 2k\pi \end{cases} \Rightarrow \begin{cases} r = 1 \\ \theta = -\frac{2\pi}{3n} + \frac{2k\pi}{n} = \frac{(6k-2)\pi}{3n} = \frac{(3k-1)2\pi}{3n} \end{cases} \quad k = 0, 1, 2, \ldots (n-1)$$

$$z'_1 = e^{-\frac{2i\pi}{3n}} \; ; \; z'_2 = e^{\frac{4i\pi}{3n}}, \ldots, \; z'_n = e^{i\frac{(3n-4)2\pi}{3n}} \text{ —— } 02\text{ pts}$$

l'ensemble des solutions est donné par :

$$\{ z_1, z_2, \ldots, z_n, z'_1, z'_2, \ldots, z'_n \}.$$